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Geodynamic thermal runaway with melting

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We consider the penetration of a solid medium by a foreign body which is large enough for frictional heating to melt the medium and maintain a thin liquid layer ahead of the body. This study is motivated by the possibility of the Earth's core having been formed by liquid iron diapirs melting their way through the solid, deformable mantle. Our principal results are the existence of a critical size for the body for the motion to be maintained under gravity and the ease with which an immiscible liquid body can penetrate at constant velocity compared to a solid one.

1. Introduction

Viscous heating is believed to play an important role in a variety of geophysical applications. For example, it is recognized that the Earth's mantle exhibits a viscous fluid behaviour on geological timescales. Thermal convection in the mantle drives plate tectonics and continental drift and the importance of viscous dissipation in this process was demonstrated by Turcotte *et al.* (1974), and Hewitt, McKenzie & Weiss (1975).

If viscous heating occurs but in addition the fluid viscosity decreases exponentially with temperature, thermal runaway can occur when a constant stress is applied (Gruntfest 1963). The viscosity of the Earth's mantle does have such a temperature dependence because the solid-state creep mechanisms responsible for the fluid behaviour are thermally activated. A number of authors have suggested that thermal runaway can occur in the Earth's mantle and that it is responsible for a substantial fraction of the observed surface volcanism (Nitsan 1973; Yuen & Schubert 1979; Melosh & Ebel 1979).

The aspect of thermal runaway with which we will be concerned is most easily illustrated by considering a Couette flow in which a plate $y = h$ is moved by a constant force parallel to a stationary plate $y = -h$. The space $-h < y < h$ is filled with a Newtonian liquid whose viscosity is

$$\mu = \mu_0 \, e^{-\beta T}, \tag{1.1}$$

where T is measured from some suitable reference temperature. If a steady flow exists, the liquid shear $\mu \, \partial u/\partial y = \tau_0$ is a constant so that the energy equation is

$$k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 = k\frac{\partial^2 T}{\partial y^2} + \frac{\tau_0^2}{\mu_0} e^{\beta T} = 0, \tag{1.2}$$

where k is the thermal conductivity, assumed constant. When the plates are isothermal so that $T = 0$ at $y = \pm h$, we find

$$e^{-\frac{1}{2}\beta T} = c \cosh \frac{\lambda y}{ch}, \tag{1.3}$$

where the real constant c satisfies $c \cosh \lambda/c = 1$ and $\lambda^2 = \beta\tau_0^2 h^2/2k\mu_0$. Hence there is no solution if

$$\beta\tau_0^2 h^2/2k\mu_0 > 0.439. \tag{1.4}$$

When this inequality is satisfied, the unsteady problem in which $\partial T/\partial t$ appears on the right-hand side of (1.2) exhibits finite-time blow-up, as in the theory of combustion. Thus an unbounded conversion of mechanical to thermal energy can occur, even though there is heat loss through the channel walls, and this is one form of thermal runaway. If the walls were thermally insulated the runaway would be more dramatic, with no steady solution existing for any value of τ_0. Equally if μ was bounded away from zero or if the flow was driven by a prescribed pressure gradient, thermal runaway could not occur (Ockendon 1979).

The mathematical and numerical analysis of flows in which the viscosity is given by (1.1) can become very difficult (Morris 1982) but the idea of relating diffusion problems with such nonlinearities to 'free boundary' problems, in which the field equations have piecewise constant coefficients, has been used in biological applications (Rinzel & Keller 1973) and combustion theory (Lacey 1981). The simple models proposed in this paper replace fluid flows in which the viscosity varies rapidly by the motion of a medium which melts at some prescribed temperature T_m from a perfectly rigid solid to a constant-viscosity liquid. Thus the Couette flow problem with isothermal walls at $T = 0$ has a steady solution in which the liquid occupies $|y| < s$ and the solid $s < |y| < h$ as long as the energy equation

$$k\frac{\partial^2 T}{\partial y^2} + \frac{\tau_0^2}{\mu_0} = 0, \quad T > T_m > 0, \quad |y| < s, \tag{1.5a}$$

has a solution satisfying

$$T = T_m, \quad \frac{\partial T}{\partial y} = \frac{-T_m}{h-s} \quad \text{on } y = s, \tag{1.5b}$$

assuming k is also the solid conductivity. It is easy to see that there are two solutions with $0 < s < h$ as long as

$$\tau_0^2 h^2/T_m k\mu_0 > 4, \tag{1.6}$$

and we now interpret thermal runaway as occurring if the shear, and hence the dissipation, is strong enough to permit the existence of a liquid region, i.e. if (1.6) holds. The mathematical advantage of this model over (1.2) is that as $\tau_0 \to \infty$, flows are possible in which $s \ll h$. The occurrence of runaway when the liquid is confined to a thin region allows lubrication theory to be employed successfully even when the geometry is more complicated, and our subsequent analysis will rely heavily on this fact.

Our aim in this paper is to consider whether thermal runaway, or a modified version of this phenomenon in which a large amount of heat is produced to change the phase but where the temperature does not significantly increase, can be the mechanism by which a solid or an immiscible fluid can melt its way through a host medium. The driving force is the differential buoyancy of the foreign body and the host medium. In the example (1.5) any imposed shear τ_0 is sufficient to permit motion if T_m is sufficiently small. However, for a buoyancy-driven finite body we will see that, with $T_m = 0$ for convenience, it is only if this buoyancy force is sufficiently large that the viscous dissipation in a thin film of molten host medium will be sufficient to provide enough heat to melt a pathway through the medium.

One possible application of this mechanism is to the problem of core formation. Iron and silicates condense at nearly the same temperature so that strongly heterogeneous accretion is highly implausible (Grossman & Larimer 1974). Thus, accepting near-homogeneous accretion, a mechanism for the later segregation of the core must be provided. Studies of the systematics of the lead isotope system indicate that the core formed within 500000 years after the Earth's formation, and is likely to have formed synchronously (Oversby & Ringwood 1971). Estimates of the energy released by the core formation indicate that it is sufficient to heat the entire Earth by about 2000 K. One explanation of core formation is that the energy of accretion and that of formation melted the entire Earth. However, most geochemists object to this hypothesis because the Earth's mantle is not fractionated and still contains significant concentrations of rare gases that should have been lost to the atmosphere had the entire Earth ever been molten (Ringwood 1975).

An alternative explanation of core segregation is the migration of bodies (diapirs) of liquid iron through the solid mantle of the Earth. During accretion the early Earth was probably covered with a magma ocean with a thickness of a few hundred kilometres (Hofmeister 1983). This magma ocean would have refined the accrescent material, with the gases forming the atmosphere and the iron sinking to the bottom. Indeed Elsasser (1963) assumed that a layer of liquid iron formed near the Earth's surface and postulated a Rayleigh–Taylor instability to explain the formation of large sinking diapirs of liquid iron. This mechanism has also been considered by Tozer (1965), Stevenson (1981), and Andrews (1982), and the solid mantle is assumed to deform by the same solid-state creep processes associated with mantle convection. This mechanism could explain the initial formation and migration of liquid iron diapirs but a simple Stokes flow calculation shows that an iron body with a diameter of 100 m would only fall 100 km in 4×10^9 years with a mantle viscosity of 10^{21} poise. However, we will show that large iron diapirs may generate sufficient energy through viscous dissipation to melt a path through the mantle at a much greater velocity than this.

Another possible application of this thermal runaway model is to magma migration. Magma must ascend from depths of 100–200 km or greater where conditions are molten, to the surface of the Earth where volcanic flows and volcanoes are observed. Mechanisms for magma migration have been reviewed by Spera (1980) and by Turcotte (1982). The role of the diapirs in magma migration has been discussed by Marsh (1978, 1982) and by Marsh & Kantha (1978). These authors have all studied the movement of liquid diapirs of prescribed shape due to the solid-state creep of the rock through which they pass. An alternative mechanism proposed here is that the heat produced by viscous dissipation can melt a path through the solid mantle. A similar mechanism was proposed by Rice (1971) to explain bulk slip, and his view is that viscous dissipation in natural convection rarely can be neglected.

In §2 of this paper we will consider the penetration of solid host rock by a solid foreign body and in §3 will consider the more realistic situation in which a liquid body melts its way through the rock. The former is less relevant geologically but is easier to describe and the results are useful for §3. Before attempting to write down any mathematical model we consider the orders of magnitude needed for the size and velocity of the foreign body if it is to be able to melt its way through relatively cool host rock under the action of viscous dissipation. We assume that the molten host rock is confined to a thin layer around the upstream (leading) surface of the foreign body and, of course, the wake. We also assume that: (i) the wake, comprising molten host rock just above the melting temperature, exerts negligible stress on the foreign body compared to the upstream lubrication forces; and (ii) real-fluid effects such as variable material properties, variable melting temperature, density changes due to melting, regelation, and surface tension are negligible. Then for a rigid foreign body we have a generalization of a problem considered by Emmons (1954). If the body is a sphere of radius a moving with a typical velocity U_0 under the action of a constant force F, and $\delta_0 \ll a$ is a typical thickness of the layer of molten host rock, then the lubrication velocity tangential to the foreign body is $O(U_0\, a/\delta_0)$ and the corresponding pressure in the layer is $O(\mu_0\, U_0\, a^2/\delta_0^3)$. Hence neglecting the wake and side-wall effects

$$F \sim \frac{\mu_0\, U_0\, a^4}{\delta_0^3}. \tag{1.7}$$

The melting of the host rock occurs as a result of heat conduction from the molten layer. In the simplest case we assume that the host rock is at, or just below, its melting temperature (taken to be zero) and that the penetrating body is at a temperature T_0 above this host rock temperature. Then with no dissipation present

$$kT_0 \sim \rho_0\, L U_0\, \delta_0, \tag{1.8}$$

where ρ_0 and L are the density and latent heat per unit mass of the host rock. Hence, for a prescribed T_0,

$$\left(\frac{F}{\mu_0}\right)^{\frac{1}{3}} \sim \frac{\rho_0\, L(aU_0)^{\frac{4}{3}}}{kT_0}. \tag{1.9}$$

However the temperature rise due to viscous dissipation in the layer must be comparable with T_0 if it is to be possible for there to be no total heat loss from the foreign body. Hence

$$a^2 \mu_0\, U_0^2 \sim kT_0\, \delta_0^2, \tag{1.10}$$

and, from (1.9), we have the obvious force balance $F \sim \rho_0\, La^2$. For a thermally insulated foreign body the temperature rise in the molten rock layer will be of order $\mu_0\, U_0^2\, a^2/k\delta_0^2$ and arbitrary values of U_0 and δ_0 are possible, provided $U_0/\delta_0^3 \sim \rho L/\mu_0\, a^2$. Thermal equilibrium can be maintained in the absence of any heat sources in the body, the average heat flow per unit area into the rock from thermal dissipation $\mu_0\, U_0^2\, a^2/\delta_0^3$ balancing the flux $\rho L U_0$ which is needed to maintain melting.

When the body rises or falls buoyantly, with a density difference $|\rho_1 - \rho_0|$, we thus see that a constant-velocity motion can only occur when the radius $a \sim \rho_0\, L/g\,|\rho_1 - \rho_0|$; larger or smaller bodies may be expected to heat or cool indefinitely. For the case of a rigid sphere of iron migrating through the mantle $\rho_0 \sim 3.3$, $\rho_1 - \rho_0 \sim 4$ and $L \sim 4 \times 10^9$ in c.g.s. units (Turcotte & Schubert 1982) so that the radius $a \sim 30$ km.

The remainder of this paper will be concerned with the quantitative analysis of this problem under assumptions (i)–(ii) and its extension to the case of immiscible liquid bodies, where the extra degree of freedom will be found to permit runaway to occur more generally.

2. Penetration by a rigid sphere

2.1. *Steady motion*

Consider first the problem of a rigid sphere moving buoyantly by melting its way through a host rock whose ambient temperature T_∞ is below its melting temperature zero. Let the sphere have radius a, a *prescribed* constant temperature $T_0 > 0$, and take

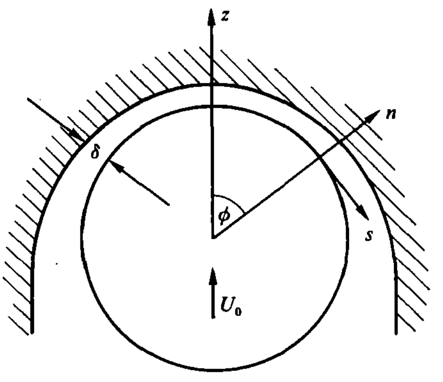

FIGURE 1. Motion of a rigid sphere.

axes fixed in the sphere moving with constant velocity in the z direction relative to the host rock (figure 1).

Now from our earlier discussion a heat balance will only be maintained in the absence of heat sources and sinks in the sphere if its velocity is $U_0 U$, where $U = O(1)$ and, from (1.8), (1.10), U_0 is defined by

$$U_0^4 = \frac{k^3 T_0^3}{\mu_0 a^2 \rho_0^2 L^2}. \tag{2.1}$$

Moreover we have seen that a typical molten-layer thickness is given by

$$\delta_0^4 = \frac{k\mu_0 a^2 T_0}{\rho^2 L^2}, \tag{2.2}$$

so that we can use hydrodynamic lubrication theory in the layer if $\delta_0 \ll a$ and also the reduced Reynolds number

$$\rho_0 \frac{U_0 \delta_0}{\mu_0} = \frac{kT_0}{L\mu_0} \ll 1. \tag{2.3}$$

We have neglected density changes during melting and so the only thermal convection which can occur is as a result of the lubrication velocity $U_0 a/\delta_0$. This effect is negligible compared to conduction across the layer if the specific heat c is such that the Stefan number is small, that is

$$\frac{cT_0}{L} \ll 1, \tag{2.4}$$

which is equivalent to the Péclet number, $c\rho_0 U_0 \delta_0/k$, being small. For the case of iron migrating through the mantle, when $T_0 \sim 100$ K and

$$a \sim 3 \times 10^6, \quad k \sim 4 \times 10^5, \quad \mu_0 \sim 1,\ c \sim 10^7$$

in c.g.s. units (Turcotte & Schubert 1982), we find $\delta_0 = (k\mu_0 T_0/\rho g^2)^{\frac{1}{4}} \sim 40$ cm, so the relevant small parameters are

$$\frac{\delta_0}{a} \sim 10^{-5}, \quad \frac{kT_0}{L\mu_0} \sim 10^{-3}, \quad \frac{cT_0}{L} \sim 10^{-1}. \tag{2.5}$$

Moreover, the large ratio of the velocity U_0 to that predicted by Stokes flow through a mantle of viscosity $\mu_\infty \sim 10^{21}$ is

$$O\left(\frac{\mu_\infty}{\mu_0}\left(\frac{\delta_0}{a}\right)^3\right) = 10^5.$$

The most important condition for our mathematical model is the melting condition

$$k\frac{\partial T}{\partial n} = -\rho_0 U U_0 L' \cos\phi, \tag{2.6}$$

where n is measured normal to the sphere and ϕ is the polar angle. We have written L' instead of L in (2.6) to account for the host medium being at some ambient temperature $T_\infty < 0$. The Péclet number in the host rock is $\rho_0 c a U_0/k$ ($\sim O(10^3)$ for the case (2.5)) and, if it is large, the host rock will adjust to its melting temperature in a thermal boundary layer whose thickness $\ll a$. The resulting heat flow out of the molten layer can then be accounted for by writing $L' = L - cT_\infty$.

Within the molten layer we define local coordinates (s, n) with n normal to the surface of the sphere and s tangential in the (z, n)-plane, where n is made non-dimensional with δ_0 defined by (2.2) and s with a so that $s = \phi$. If (u, v) are corresponding local velocity components, made non-dimensional with aU_0/δ_0 and U_0 respectively where U_0 is defined by (2.1), the boundary conditions on the molten layer are

$$u = v = 0, \quad T = 1 \quad (n = 0), \tag{2.7}$$

$$\left.\begin{aligned} &v = -U\cos\phi, \quad u = O\left(\frac{\delta_0}{a}\right), &(2.8)\\ &T = 0, \quad \frac{\partial T}{\partial n} = -U\cos\phi &(2.9)\end{aligned}\right\} \quad (n = \delta(\phi)).$$

Here T has been made non-dimensional with T_0, and we have again assumed no density change on the melting boundary. Following lubrication theory, we neglect terms of $O(\delta_0/a)$ and the viscous-flow equations in the molten layer reduce to

$$\frac{\partial p}{\partial \phi} = \frac{\partial^2 u}{\partial n^2}, \quad \frac{\partial p}{\partial n} = 0, \tag{2.10}$$

together with the continuity equation, in spherical polars,

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}(u\sin\phi) + \frac{\partial v}{\partial n} = 0, \tag{2.11}$$

where p is the pressure, including hydrostatic pressure, made dimensionless with $\mu U_0 a^2/\delta_0^3$.

The energy equation in the molten layer in this approximation reduces to

$$\frac{\partial^2 T}{\partial n^2} + \left(\frac{\partial u}{\partial n}\right)^2 = 0. \tag{2.12}$$

Integrating equation (2.10), $u = \frac{1}{2}n(n-\delta)\,\partial p/\partial\phi$. From (2.11), using boundary conditions (2.7), (2.8) and the condition that $\partial p/\partial \phi$ is bounded at $\phi = 0$,

$$\delta^3\frac{\partial p}{\partial \phi} = -6U\sin\phi. \tag{2.13}$$

Integrating equation (2.12),

$$T=-\frac{1}{12}\left(\frac{\partial p}{\partial \phi}\right)^2\left\{\left(n-\frac{\delta}{2}\right)^4-\left(\frac{\delta}{2}\right)^4\right\}+1-\frac{n}{\delta}, \tag{2.14}$$

and applying the Stefan condition (2.9) on $n = \delta$,

$$U\cos\phi = \tfrac{1}{24}\delta^3\left(\frac{\partial p}{\partial \phi}\right)^2+\frac{1}{\delta}. \tag{2.15}$$

Finally eliminating $\partial p/\partial\phi$ from (2.13), δ is given by

$$U\delta^3\cos\phi-\delta^2-\tfrac{3}{2}U^2\sin^2\phi = 0. \tag{2.16}$$

This cubic for δ always has a positive root for $U > 0$, $0 \leqslant \phi < \frac{1}{2}\pi$, and $\delta \sim 1/U$ as $\phi \to 0$; $\delta \sim \sec\phi/U$ as $\phi \to \frac{1}{2}\pi$. The existence of this root for δ implies that thermal runaway can occur in the sense discussed in the introduction.

If the sphere has a high thermal conductivity, its temperature will remain constant in the absence of internal heat sources or sinks when the net heat flux across its surface is zero. Neglecting heat losses into the wake from the lower half of the sphere, this condition is

$$\int_0^{\frac{1}{2}\pi}\left(\frac{\partial T}{\partial n}\right)_{n=0}\sin\phi\,\mathrm{d}\phi = 0.$$

Using (2.14)–(2.16) this condition may be reduced to

$$\int_0^{\frac{1}{2}\pi}\frac{\sin\phi}{\delta(\phi)}\,\mathrm{d}\phi = \tfrac{1}{4}U, \tag{2.17}$$

where $\delta(\phi)$ is given by (2.16). This expression determines the value of $U \simeq 1.22$ for which a steady motion with constant temperature is possible.

For the sphere velocity to remain constant the total forces on the sphere must also be in equilibrium. If we assume that there is no force on the lower half of the sphere due to the wake, then the buoyancy force must balance the pressure force integrated over the upper hemisphere, so that

$$\int_0^{\frac{1}{2}\pi} p\cos\phi\sin\phi\,\mathrm{d}\phi = \frac{|\rho_1-\rho_0|\,g\delta_0^3}{3\mu_0 U_0 a}, \tag{2.18}$$

where ρ_1 is the density of the sphere, which is less than ρ_0 if g is in the negative z direction. Defining

$$\gamma = \frac{|\rho_1-\rho_0|}{\rho_0}\frac{ag}{L}, \tag{2.19}$$

and using (2.13), equation (2.18) may be written

$$\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\phi'}\frac{6\sin\phi}{(\delta(\phi))^3}\mathrm{d}\phi\cos\phi'\sin\phi'\,\mathrm{d}\phi' = \frac{-\gamma}{3U},$$

where it has been assumed that $p = 0$ at $\phi = \frac{1}{2}\pi$, where the lubrication layer meets the wake. This simplifies to

$$\int_0^{\frac{1}{2}\pi}\left(\frac{\sin\phi}{\delta(\phi)}\right)^3\mathrm{d}\phi = \frac{\gamma}{9U}, \tag{2.20}$$

and, using (2.16) and (2.17), $\gamma = \frac{3}{2}$, which is just the global energy balance for the sphere.

A steady motion in mechanical and thermal equilibrium is therefore possible if U and γ take specific values. This implies that the velocity if proportional to $T_0^{\frac{3}{4}}$ and the radius of the sphere is prescribed, with a value of about 20 km in the case (2.5).

2.2. *Unsteady motion*

We may speculate that, if $\gamma > \frac{3}{2}$, the sphere will accelerate and its temperature will increase without bound, while if $\gamma < \frac{3}{2}$ it will be brought to rest by solidification of the molten layer. Rather than carry out a full unsteady analysis, we assume that the timescales associated with the mechanical inertia of the sphere and with the phase change are short compared with that of the heat capacity of the sphere. The sphere velocity and temperature are now $U_0\, U(t)$ and $T_0\, \theta(t)$ respectively, where U_0 is defined by (2.1), and time only enters as a parameter in the lubrication equations. Hence (2.16) becomes

$$U\delta^3 \cos\phi - \theta\delta^2 - \tfrac{3}{2}U^2 \sin^2\phi = 0 \tag{2.21}$$

and, in the absence of mechanical inertia, the force balance (2.20) is

$$\gamma = 9U \int_0^{\frac{1}{2}\pi} \left(\frac{\sin\phi}{\delta}\right)^3 \mathrm{d}\phi. \tag{2.22}$$

If we denote the heat capacity of the sphere by α, the heat balance (2.17) is now

$$\alpha \frac{\mathrm{d}\theta}{\mathrm{d}t} = \tfrac{1}{3}\pi U(\gamma - \tfrac{3}{2}). \tag{2.23}$$

Equations (2.22) and (2.23) may be rescaled conveniently by the transformation $\delta = \theta^{\frac{1}{4}}\bar{\delta}(\phi)$, $U = \theta^{\frac{3}{4}}\overline{U}$, and θ only appears explicitly in the heat-balance equation

$$4\alpha \frac{\mathrm{d}}{\mathrm{d}t}(\theta^{\frac{1}{4}}) = \tfrac{1}{3}\pi \overline{U}(\gamma - \tfrac{3}{2}). \tag{2.24}$$

Hence θ is monotonic, increasing or decreasing to zero in finite time depending on $\operatorname{sgn}(\gamma - \frac{3}{2})$, and the steady motion with $\gamma = \frac{3}{2}$ will not be stable to small changes in γ. Moreover this model gives no terminal velocity as might have been expected by analogy with other motions in resistive media.

When $\gamma = \frac{3}{2}$, θ can take any constant value and $U\theta^{-\frac{3}{4}}$ is the steady velocity defined by (2.16) and (2.17). All the possible steady motions are parametrized by the sphere temperature, in accordance with our original non-dimensionalization.

3. Penetration by an immiscible liquid body

3.1. *General unsteady motion*

Changing the shape of the rigid body from a sphere only alters constants in the equations and leads to a result similar to (2.24), namely that only one critical value of a size parameter γ is possible for steady motion. However, when we consider the penetrating region to be an immiscible liquid body with negligible surface tension, the predictions of the model are significantly different.

We again assume the Stefan number is small, where a^3 is now the volume of the body and a is large compared to δ_0. The velocity induced in the body is $O(U_0 a/\delta_0)$

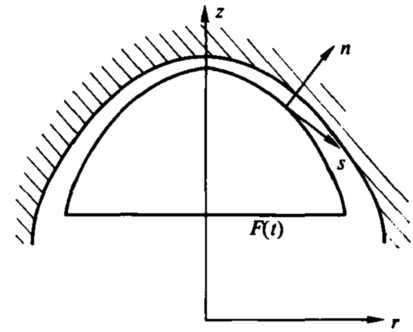

FIGURE 2. Motion of an immiscible liquid body.

and we also assume the dynamic viscosity μ_1 of the body is such that $\rho_1 ga\delta_0 \gg U_0\mu_1$ and $g\delta_0^2 \gg aU_0^2$, so that the pressure in the drop is hydrostatic. This is consistent with our assumptions about orders of magnitude in §2 if $\mu_1 \sim \mu_0$. However, the fact that the pressure in the body is hydrostatic means we will be able to impose a local force balance on the molten layer, which will be crucial in calculating its shape. In addition to continuity of pressure at the body surface, we will also impose continuity of normal velocity and zero shear stress since pressure forces dominate shear forces in lubrication theory.

Assuming the body density $\rho_1 \sim \rho_0$, δ_0 will be such that the hydrostatic and lubrication pressures are comparable, so $\rho_0 ga \sim \mu_0 U_0 a^2/\delta_0^3$ in addition to (2.4), and we define γ as before by (2.19).

We describe an axisymmetric body as in figure 2 by $z = f(r,t)$ with respect to axes fixed in space; the base of the body is assumed planar at $z = F(t)$, with the wake in $z < F(t)$. Local coordinates (s,n) are chosen with moving origin in the body surface at a typical point (r,f) with s tangential and n normal to the surface and a typical velocity U_0 is defined from (2.1) as before.

The lubrication problem corresponding to (2.7), (2.8) and (2.9) becomes

$$v = 0 = \frac{\partial u}{\partial n}, \quad T = \theta(t), \quad p = \gamma f \quad (n=0); \tag{3.1}$$

$$\left.\begin{aligned} u = 0, \quad v &= -\frac{\partial f}{\partial t}\frac{\partial r}{\partial s}, \qquad (3.2)\\ T = 0, \quad \frac{\partial T}{\partial n} &= -\frac{\partial f}{\partial t}\frac{\partial r}{\partial s} \qquad (3.3) \end{aligned}\right\} (n=\delta);$$

together with

$$\frac{\partial p}{\partial s} = \frac{\partial^2 u}{\partial n^2}, \quad \frac{\partial p}{\partial n} = 0, \quad \frac{1}{r}\frac{\partial}{\partial s}(ru) + \frac{\partial r}{\partial n} = 0 \tag{3.4}$$

and

$$\frac{\partial^2 T}{\partial n^2} + \left(\frac{\partial u}{\partial n}\right)^2 = 0. \tag{3.5}$$

Note that in steady conditions $\partial f/\partial t$ is constant, and that in the notation of the problem for the sphere $r = \sin\phi$, with $\partial r/\partial s = \cos\phi$ and $\partial f/\partial s = \sin\phi$.

Integrating equations (3.4) and (3.5) gives

$$r\frac{\partial f}{\partial t} = -\tfrac{1}{3}\gamma\frac{\partial}{\partial r}\left(r\delta^3\frac{\partial f}{\partial s}\right) \tag{3.6}$$

and
$$T = -\frac{\gamma^2}{12}(n^4-\delta^3 n)\left(\frac{\partial f}{\partial s}\right)^2 + \theta\left(1-\frac{n}{\delta}\right).$$

Applying the Stefan condition (3.3)
$$\frac{\gamma^2\delta^3}{4}\left(\frac{\partial f}{\partial s}\right)^2 + \frac{\theta}{\delta} = \frac{\partial f}{\partial t}\frac{\partial r}{\partial s}. \tag{3.7}$$

In the case of the rigid sphere, (3.6) simplified to (2.13) and explicitly gave the pressure gradient; (3.7) is a statement equivalent to (2.15). However for an unsteady liquid body it is no longer possible to eliminate δ or f between these two equations. They may however be simplified by an appropriate scaling similar to that used in deriving (2.24) and we make the transformation
$$\delta = \left(\frac{3\theta}{2\gamma}\right)^{\frac{1}{4}}\bar{\delta}, \quad \frac{\partial}{\partial t} = \theta^{\frac{3}{4}}\left(\frac{2\gamma}{3}\right)^{\frac{1}{4}}\frac{\partial}{\partial \bar{t}}, \quad x = \frac{r^2}{2}, \tag{3.8}$$

so that
$$\frac{\partial f}{\partial \bar{t}} = -\frac{\partial}{\partial x}\left\{\frac{x\bar{\delta}^3\dfrac{\partial f}{\partial x}}{\left(1+2x\left(\dfrac{\partial f}{\partial x}\right)^2\right)^{\frac{1}{2}}}\right\}, \tag{3.9}$$

and
$$\frac{\partial f}{\partial \bar{t}} = \frac{3\gamma}{4}\frac{x\bar{\delta}^3(\partial f/\partial x)^2}{(1+2x(\partial f/\partial x)^2)^{\frac{1}{2}}} + \frac{(1+2x(\partial f/\partial x)^2)^{\frac{1}{2}}}{\bar{\delta}}. \tag{3.10}$$

Three global conservation laws must be applied to the body; we have already chosen a so that its volume is unity and we assume that its non-dimensional constant momentum in the z direction is U^*. We recall that the immiscible liquid body is at hydrostatic pressure and is assumed to have a high thermal conductivity. Then from conservation of volume
$$-\int_0^{X(\bar{t})} x\frac{\partial f}{\partial x}\,dx = \frac{1}{2\pi} = \int_0^{X(\bar{t})}(f-F)\,dx, \tag{3.11}$$

where $2\pi X(\bar{t})$ is the area of the base of the body $z = F(\bar{t})$. From conservation of momentum
$$-\frac{\partial}{\partial t}\int_0^{X(\bar{t})} xf\frac{\partial f}{\partial x}\,dx = \frac{U^*}{2\pi},$$

so that
$$\theta^{\frac{3}{4}}\frac{\partial}{\partial \bar{t}}\int_0^{X(\bar{t})} xf\frac{\partial f}{\partial x}\,dx = -\frac{\bar{U}}{2\pi}, \tag{3.12}$$

where
$$U^* = \left(\frac{2\gamma}{3}\right)^{\frac{1}{4}}\bar{U}.$$

From conservation of heat
$$\frac{\alpha}{2\pi}\frac{d\theta}{d\bar{t}} = \int_0^{X(\bar{t})}\left\{\frac{\gamma x\bar{\delta}^3(\partial f/\partial x)^2}{(1+2x(\partial f/\partial x)^2)^{\frac{1}{2}}} - \frac{\partial f}{\partial \bar{t}}\right\}dx. \tag{3.13}$$

3.2. *Steady motion*

An exact solution of these equations may be found for a body of constant shape in the form
$$f = \bar{U}\bar{t} - g(x), \quad \theta = 1. \tag{3.14}$$

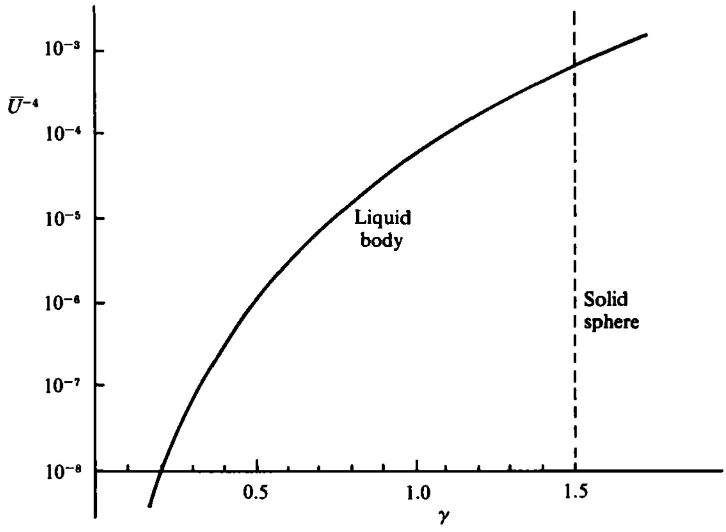

FIGURE 3. Velocity and size parameters for steady motion.

Equation (3.9) may be integrated, and together with (3.10) and the condition that $\bar{\delta}^3\, \partial f/\partial x$ is bounded at $x = 0$, gives

$$\bar{U}^4(1-\tfrac{3}{4}\gamma xh)^3 = h(1+2xh^2), \tag{3.15}$$

where $h = \mathrm{d}g/\mathrm{d}x$. Equations (3.11) and (3.12) are now identical and (3.13) is also satisfied if $X(\bar{t}) = X_0 = \gamma/2\pi$ and

$$\int_0^{\gamma/2\pi} xh\,\mathrm{d}x = \frac{1}{2\pi}. \tag{3.16}$$

Equations (3.15) describes the shape of the body in this steady situation and its nose radius of curvature is $\bar{U}^4$. Such shape will only be possible if condition (3.16) is also satisfied, that is there is a functional relation between γ the size parameter and $\bar{U}$ the non-dimensional momentum.

From (3.15) it is easy to see that $\mathrm{d}(hx)/\mathrm{d}x > 0$ for $x, h > 0$ and that $hx \sim 4/3\gamma$ as $x \to +\infty$. Hence hx is a monotone increasing function of x and for any fixed γ the left-hand side of (3.16) tends to

$$\frac{4}{3\gamma}\frac{\gamma}{2\pi} = \frac{2}{3\pi} \quad \text{as } \bar{U} \to \infty$$

and zero as $\bar{U} \to 0$. Hence, by continuity, there is at least one value of $\bar{U}$ satisfying (3.16) for any γ. Moreover, as $\gamma \to 0$ or ∞, $x = O(\gamma)$, $h = O(\gamma^{-2})$ so, from (3.15), $\bar{U}^4 \sim \gamma^{-5}$ as $\gamma \to 0$ and $\bar{U}^4 \sim \gamma^{-2}$ as $\gamma \to \infty$. The relation between $\bar{U}$ and γ is shown in figure 3; we can again think of the body temperature as parametrizing points on this curve. Unlike the situation for a solid body, there is a continuum of liquid body shapes depending on their size γ and typical profiles are shown in figure 4.

3.3. *Unsteady motion*

We may regard (3.10) as an algebraic equation for $\bar{\delta}$ and substitute into (3.9) to obtain a quasilinear hyperbolic partial differential equation for f, one of its characteristics being that $\bar{t}$ = const; $x = 0$ is a singular line on which the characteristics coincide.

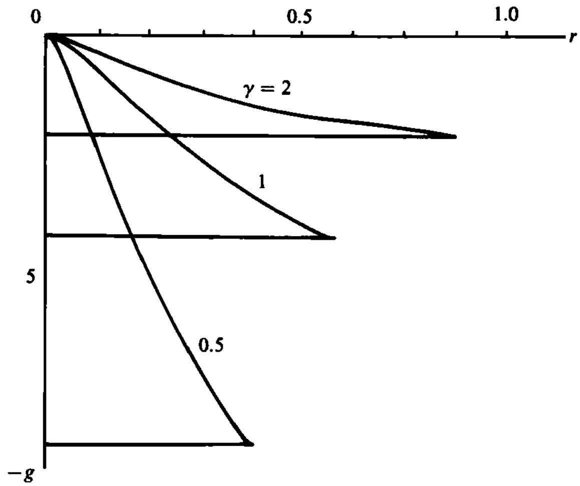

FIGURE 4. Liquid body shapes.

Hence we may expect that local wavelike solutions are possible away from the nose of the body but even the linear stability analysis of the steady motions derived above is very complicated. However, we note that a body with prescribed volume containing heat sources or sinks could move steadily, conserving mass and momentum, with f given by (3.14) and

$$\int_0^{X_0} xh\,\mathrm{d}x = \frac{1}{2\pi},$$

where, again, $h = \mathrm{d}g/\mathrm{d}x$. However, from (3.13) its temperature would satisfy

$$\frac{\alpha}{\alpha\pi}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\bar{t}} = \bar{U}\left(\frac{\gamma}{2\pi} - X_0\right), \tag{3.17}$$

so that a slender body with $X_0 < \gamma/2\pi$ would require a heat sink and a blunt body would require a heat source to maintain the motion. We note from (3.15) that, with γ fixed, an increase in $\bar{U}$ corresponds to an increase in h and hence, from (3.16), a decrease in X_0, so that the more slender bodies have greater velocities.

4. Conclusion

We have been able to carry out a fairly complete analysis of the motion of a solid body melting its way through a host medium. Under the action of a constant force, a continuum of steady velocities and temperatures is possible with the fourth power of the velocity being proportional to the cube of the temperature, but these steady conditions can only be attained under the action of gravity if the solid has precisely the correct size.

The problem of penetration by an immiscible liquid body is more complicated, although it seems that steady motion is possible for bodies with a range of sizes, all of which are of the same order of magnitude and each with its own velocity and temperature. Moreover the more slender the diapir the more rapidly it penetrates, which is the opposite result to that obtained from Stokes law, a conclusion implicit in the analysis of Morris (1982). Whether these motions are stable or not can probably be decided only by numerical integration of the hyperbolic system (3.9), (3.10), which is singular at the nose $x = 0$.

From this analysis we argue that long, thin, liquid iron bodies can melt their way to the core. Each diapir will have a wake of molten mantle rock which is cooled by the surrounding rock and eventually solidifies, the heat being lost at the Earth's surface. This is a heat-pipe mechanism that transports the heat of core formation to the Earth's surface without melting the entire mantle.

Our analysis in § 3 may also be relevant for the upward migration of magma diapirs, for which the viscosity is comparable to that of liquid iron. However, the density difference is smaller than that for liquid iron so there is less energy available for the viscous dissipation process. Also, some modification to our assumptions about the wake are needed to take account of the feeder pipe which is known to occur (Ribe 1983).

Finally, there is some similarity between the problem described in §2 and that of the rise of hot mantle diapirs through the mantle. Here the viscosity variation is much smaller but its strong temperature dependence has been suggested as a mechanism for the formation of thin thermal plumes (Loper & Stacey 1983).

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